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MAPPINGS OF BAR CONSTRUCTIONS

BY
JEAN-PIERRE MEYER

ABSTRACT
Quillen's famous Theorem B describes the homotopy fiber of $BF : B\mathscr{C} \to B\mathscr{C}'$, where $f : \mathscr{C} \to \mathscr{C}'$ is a functor and B the classifying space functor. This is here generalized to a description of the homotopy fiber of

$$B(F, \alpha, \beta) : B(Y, \mathscr{C}, X) \to B(Y', \mathscr{C}', X'),$$

where $(F, \alpha, \beta) : (Y, \mathscr{C}, X) \to (Y', \mathscr{C}', X')$ is a mapping of 2-sided bar construction data.

The purpose of this note is to present a generalization of Quillen's Theorems A and B, [8], to bar constructions. In the spirit of [3], we would like to consider morphisms of bar constructions in as general a setting as possible. For the sake of brevity, however, we restrict ourselves here to the classical topological case. This note may serve as background and motivation for the more general treatment in [4]; it may suffice for those interested strictly in the topological case.

1. Bar constructions

We recall briefly certain definitions:

(i) The *nerve* of a category $\mathscr{C}$ is the simplicial set $N_*(\mathscr{C})$ whose n-simplices are sequences

$$u : A_0 \xrightarrow{a_1} A_1 \to \cdots \xrightarrow{a_n} A_n$$

of n composable arrows. The simplicial operators are defined by composition and insertion of identity arrows, [9]. If $\mathscr{C}$ is a topological category, then $N_*(\mathscr{C})$ is a simplicial space.

(ii) If $\mathscr{C}$ is a category and $X : \mathscr{C} \to \mathrm{Top}$, $Y : \mathscr{C}^* \to \mathrm{Top}$ are functors, then the

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simplicial bar construction $B_*(Y, \mathscr{C}, X)$ or $B_*(\mathscr{C}, Y \times X)$ is the simplicial space whose n-component is given by

$$B_n(Y, \mathscr{C}, X) = \coprod_u Y(A_n) \times X(A_0).$$

It is useful to think of this definition as attaching to the n-simplex u of $N_n(\mathscr{C})$ the "coefficient space" Y(target of u)$\times X$(source of u). Two special cases of interest are: (α) $\mathscr{C}$ corresponds to a group G; then X corresponds to a left G-space, Y to a right G-space and $B_n = \{(y, g_1, \dots, g_n, x)\}$; ($\beta$) $Y = *$, then $B_*(*, \mathscr{C}, X) = \operatorname{hocolim}_* X$.

If we wish to handle the situation where $\mathscr{C}$ is a topological category and X, Y are continuous, it is best to use the language of $\mathscr{O}$-graphs, [2], as in (iii) below.

(iii) If $\mathscr{G}$ is a monoid in the category of $\mathscr{O}$-graphs, $\mathscr{Y}$ a right $\mathscr{O}$-graph over $\mathscr{G}$, $\mathscr{X}$ a left $\mathscr{O}$-graph over $\mathscr{G}$, then the *simplicial bar construction* $B_*(\mathscr{Y}, \mathscr{G}, \mathscr{X})$ is the simplicial space with $B_n(\mathscr{Y}, \mathscr{G}, \mathscr{X}) = \mathscr{Y} \square \mathscr{G} \square \cdots \square \mathscr{G} \square \mathscr{X}$ (n copies of $\mathscr{G}$). We can again write points of $B_n(\mathscr{Y}, \mathscr{G}, \mathscr{X})$ as (y, u, x) or $(y, a_1, a_2, \dots, a_n, x)$.

(iv) In order to define mappings of bar-constructions, we need to have morphisms of the corresponding data. For (i), this is simply a functor $f : \mathscr{C} \to \mathscr{C}'$. For (ii), this is a functor $f : \mathscr{C} \to \mathscr{C}'$ and natural transformations $\alpha : Y \Rightarrow Y'f$, $\beta : X \Rightarrow X'f$. For (iii), this is a morphism of graph monoids ($\mathscr{O}$, $\mathscr{O}'$ need not be identical) $f : \mathscr{G} \to \mathscr{G}'$ and f-morphisms $\alpha : \mathscr{Y} \to \mathscr{Y}'$, $\beta : \mathscr{X} \to \mathscr{X}'$. If is clear how to define such f-morphisms. It is also easy to see that morphisms, as given above, induce simplicial mappings of the corresponding simplicial bar-constructions.

Applying the geometric realization functor, we obtain the *classifying space* $B\mathscr{C}$ in (i), and the *bar constructions* $B(Y, \mathscr{C}, X)$ in (ii) (with special cases (α) $B(Y, G, X)$, (β) $\operatorname{hocolim} X$), $B(\mathscr{Y}, \mathscr{G}, \mathscr{X})$ in (iii). Also, from (iv) we obtain $Bf : B\mathscr{C} \to B\mathscr{C}'$, $B(f, \alpha, \beta) : B(Y, \mathscr{C}, X) \to B(Y', \mathscr{C}', X')$ and $B(f, \alpha, \beta) : B(\mathscr{Y}, \mathscr{C}, \mathscr{X}) \to B(\mathscr{Y}', \mathscr{C}', \mathscr{X}')$. Our aim is to study $B(f, \alpha, \beta)$.

2. Known theorems

In this section, we recall some known theorems on mappings of bar-constructions.

(i) Let $f : \mathscr{C} \to \mathscr{C}'$ be a functor and Y an object of $\mathscr{C}'$. The comma category ${}_Y\backslash f$ has objects (X, v) with X an object of $\mathscr{C}$ and $v : Y \to fX$; morphisms from (X, v) to (X', v') are arrows $w : X \to X'$ such that $fw \cdot v = v'$. Then Quillen, [8], proves:

THEOREM A. *If $B({}_Y\backslash f)$ is contractible, then $Bf : B\mathscr{C} \to B\mathscr{C}'$ is a homotopy equivalence.*

THEOREM B. *If, for all arrows $Y \to Y'$ in $\mathscr{C}'$, the induced map $B({}_{Y'}\backslash f) \to B({}_Y\backslash f)$ is a homotopy equivalence, then $Bf : B\mathscr{C} \to B\mathscr{C}'$ has homotopy fiber $B({}_Y\backslash f)$.*

(ii) hocolims and their duals, holims, can be defined for functors into simplicial sets ($\mathscr{S}$) rather than Top, and Bousfield–Kan ([1], XI, 9.2) prove:

COFINALITY THEOREM. *If $f : \mathscr{C} \to \mathscr{C}'$ is left cofinal, $X : \mathscr{C} \to \mathscr{S}$, $X' : \mathscr{C}' \to \mathscr{S}$ and $X'f = X$, then* $\operatorname{holim} f : \operatorname{holim} X' \to \operatorname{holim} X$ *is a homotopy-equivalence, if each $X'(C')$, $C' \in \mathscr{C}'$, is fibrant.*

(iii) In the classical case, where G is a group or a group-like monoid, Milnor, [5], Stasheff, [10], May, [2] and many other authors prove the following, or variations of them:

$B(*, G, G) \to B(*, G, *) = BG$ is a quasi-fibration with fiber G,

$B(Y, G, X) \to B(*, G, X)$ is a quasi-fibration with fiber Y,

$B(Y, G, X) \to B(Y, G, *)$ is a quasi-fibration with fiber X.

(iv) Let $f : H \to G$ be a map of group-like topological monoids. Then $Bf : BH \to BG$ is a quasi-fibration with fiber $B(G, H, *)$; see Stasheff [10], and May [2].

3. The generalization

In [4], we will generalize the above theorems in 3 ways:

(a) arbitrary bar-construction morphisms (f, α, β) will be considered,
(b) Top or $\mathscr{S}$ will be replaced by more general categories with homotopy,
(c) homotopy equivalences or quasi-fibrations will be replaced by more general families of morphisms.

In this note, we deal only with (a), and generalize Quillen's approach to bar-constructions in Top.

Recall that Quillen's proof of Theorem B, with a slight variation due to Thomason, [10], consists of 3 steps:

(1) He defines a bisimplicial set $S(f)_{**}$, with bisimplicial maps

$$\pi^h : S(f)_{**} \to {}^{*\times}B_*(\mathscr{C}),$$

$$\pi^v : S(f)_{**} \to B_*(\mathscr{C}')_{\times *},$$

and a simplicial homotopy

$$\begin{array}{ccc} & DS(f)_{**} & \\ {}^{\pi^h/D}\swarrow & \simeq & \searrow^{\pi^v/D} \\ N_*(\mathscr{C}) & \xrightarrow{N_*f} & N_*(\mathscr{C}') \end{array}$$

(2) He proves that the geometric realization of π^h is a homotopy equivalence.

(3) He proves that the geometric realization of π^v is a quasi-fibration with fiber $B({}_Y\backslash f)$.

We now give a few more details and adapt Quillen's approach to bar-constructions.

Let

$$u \in N_q(\mathscr{C}), \qquad u = (A_0 \xrightarrow{a_1} A_1 \to \cdots \xrightarrow{a_q} A_q),$$

$$v \in N_p(\mathscr{C}'), \qquad v = (B_0 \xrightarrow{b_1} B_1 \to \cdots \xrightarrow{b_p} B_p),$$

$$w : B_p \to fA_0.$$

Then $S(f)_{pq} = \{(u, v, w)\}$, with obvious bisimplicial operators and projections π^h, π^v. We wish to define a bisimplicial set $T_{**} = T(f, \alpha, \beta)_{*\times}$ together with

$$\pi^h : T_{**} \to {}^{*\times}B_*(Y, \mathscr{C}, X),$$

$$\pi^v : T_{**} \to B_*(Y', \mathscr{C}', X')_{\times *},$$

and a simplicial homotopy

$$\begin{array}{ccc} & DT(f, \alpha, \beta)_{**} & \\ {}^{\pi^h/D}\swarrow & \simeq & \searrow^{\pi^v/D} \\ B_*(Y, \mathscr{C}, X) & \xrightarrow[B_*(f, \alpha, \beta)]{} & B_*(Y', \mathscr{C}', X') \end{array}$$

It seems natural to try to define T_{**} by attaching coefficient spaces to each $(u, v, w) \in S_{pq}(f)$. Let $(\bar{x}, \bar{y})$ be a point of such a coefficient space. In order to define π^h, π^v, we need mappings

$$\bar{x} \mapsto x \in X(A_0), \quad \bar{x} \mapsto x' \in X'(B_0), \quad \bar{y} \mapsto y \in Y(A_q), \quad \bar{y} \mapsto y' \in Y'(B_p).$$

Since our data yield

$$Y(A_q) \xrightarrow{\beta} Y'(fA_q) \xrightarrow{Y'(fu)} Y'(fA_0) \xrightarrow{Y'(w)} Y'(B_p),$$

we may take $\bar{y} = y$ and the Y-component of our coefficient space to be $Y(A_q)$. On the other hand, the only connection between x, x' given by our data is

$$\begin{array}{ccccc} & & & & X(A_0) \\ & & & & \big\downarrow \alpha \\ X'(B_0) & \xrightarrow{X'(u)} & X'(B_p) & \xrightarrow{X'(v)} & X'(fA_0) \end{array}$$

Thus we take the X-component of our coefficient space to be the pullback of this diagram, denoted by $X'(B_0) \nabla X(A_0)$, and define

$$T_{pq}(f, \alpha, \beta) = \coprod_{(u,v,w)} Y(A_q) \times (X'(B_0) \nabla X(A_0)).$$

It is easy to define bisimplicial operators and projections π^h, π^v, as required. Let

$$j(u, v, w): Y(A_q) \times (X'(B_0) \nabla X(A_0)) \to T_{pq},$$

$$j(v): Y'(B_q) \times X'(B_0) \to B_q(Y', \mathscr{C}', X')$$

be the canonical inclusions; let $p = q = n$,

$$v(i): B_0 \xrightarrow{b_1} \cdots \xrightarrow{b_i} B_i \xrightarrow{fa_i \cdots fa_1 \cdot w \cdot b_n \cdots b_{i+1}} fA_i \to \cdots \xrightarrow{fa_n} fA_n$$

and $\pi: X'(B_0) \nabla X(A_0) \to X'(B_0)$. Then the simplicial homotopy is given by

$$h_i \cdot j(u, v, w) = j(v(i)) \cdot [\beta \times \pi].$$

This takes care of the first step (1).

Consider now π^h and realize it geometrically in the p-direction; for fixed A_0, the set of all (v, w) is the set of simplices of $N_*(\mathscr{C}'/fA_0)$. Define

$\bar{X}_{A_0}: \mathscr{C}'/fA_0 \to \text{Top}$ by $\bar{X}_{A_0}(B \to fA_0) = X'(B) \nabla X(A_0)$. Then

$$T_{pq} = B_q(\mathscr{C}, Y \times B_p(*, \mathscr{C}'/f -, \bar{X}_-))$$

and

$$R^h \pi^h : B_q(\mathscr{C}, Y \times B(*, \mathscr{C}'/f -, \bar{X}_-)) \to B_q(\mathscr{C}, Y \times X).$$

The category $\mathscr{C}'/fA$ has the terminal object $fA \xrightarrow{1} fA$ and so, by [1], XII, 3.1, or [3], 7.5,

$$B(*, \mathscr{C}'/fA, \bar{X}_A) = \text{hocolim}\, \bar{X}_A \to \bar{X}_A(fA \xrightarrow{1} fA) = X(A)$$

is a homotopy equivalence. Hence by the May–Tornehave theorem,

$$R\pi^h = R^v R^h \pi^h : RT_{**} \to B(\mathscr{C}, Y \times X)$$

is a homotopy equivalence. This is the second step (2).

Consider now π^v and realize it geometrically in the q-direction; the category ${}_B\backslash f$ has objects of the form $(A, x : B \to fA)$. Define $\bar{Y} : ({}_B\backslash f)^* \to \text{Top}$ by $(A, B \xrightarrow{x} FA) \mapsto Y(A)$, $\bar{X}_b : {}_B\backslash f \to \text{Top}$ sending (A, x) to the pullback of

$$\begin{array}{ccc} & & X(A) \\ & & \downarrow{\scriptstyle \alpha} \\ X'(B') \to & X'(B) \to & X'(fA) \end{array}$$

This is defined *only if* we have $B' \xrightarrow{b} B$. Thus, for any $B_0 \to B_p$ and, so for any $v \in N_p(\mathscr{C}')$, the space $B({}_{B_p}\backslash f, \bar{Y} \times \bar{X}_v)$ is defined; the function $v \mapsto B({}_{B_p}\backslash f, \bar{Y} \times \bar{X}_v)$ is sufficient, it turns out (see appendix), to define $B_*(\mathscr{C}', B({}_-\backslash f, \bar{Y} \times \bar{X}_=))$ and $R^v \pi^v$ is

$$R^v \pi^v : B_p(\mathscr{C}', B({}_-\backslash f, \bar{Y} \times \bar{X}_=)) \to B_p(\mathscr{C}', Y' \times X').$$

Applying the argument for the lemma, p. 90, [8], or [7], Theorem, or [3], 3.13 and 3.14, we obtain:

MAIN THEOREM. *Let* $(f, \alpha, \beta) : (Y, \mathscr{C}, X) \to (Y', \mathscr{C}', X')$. *Assume*:

(i) $N_*(\mathscr{C})$, $N_*(\mathscr{C}')$ *are good simplicial spaces.*

(ii) *For all* $b : B' \to B$, $B({}_B\backslash f, \bar{Y} \times \bar{X}_b) \to Y'(B) \times X'(B')$ *has homotopy fiber* F.

(iii) *For all* $v : B_0 \to B_1 \to \cdots \to B_p$, *the squares below induce homotopy equivalences on the homotopy fibers*:

$$\begin{array}{ccccc} B({}_{B_{p-1}}\backslash f, \bar{Y} \times \bar{X}_{d_p v}) & \leftarrow & B({}_{B_p}\backslash f, \bar{Y} \times \bar{X}_v) & \rightarrow & B({}_{B_p}\backslash f, \bar{Y} \times \bar{X}_{d_0 v}) \\ \downarrow & & \downarrow & & \downarrow \\ Y'(B_{p-1}) \times X'(B_0) & \leftarrow & Y'(B_p) \times X'(B_0) & \rightarrow & Y'(B_p) \times X'(B_1) \end{array}$$

Then $B(f, \alpha, \beta)$ *also has homotopy fiber* F.

It is not difficult to see that (iii) can be simplified to:

(iii)′ *For all* $b : B' \to B$, *the squares below induce homotopy equivalences on the homotopy fibers*:

$$\begin{array}{ccccc} B({}_B\backslash f, \bar{Y} \times \bar{X}_1) & \leftarrow & B({}_B\backslash f, \bar{Y} \times \bar{X}_b) & \rightarrow & B({}_{B'}\backslash f, \bar{Y} \times \bar{X}_1) \\ \downarrow & & \downarrow & & \downarrow \\ Y'(B) \times X'(B) & \leftarrow & Y'(B) \times X'(B') & \rightarrow & Y'(B') \times X'(B') \end{array}$$

Furthermore, the left square is easily seen to be a pullback; however, pullbacks do not necessarily preserve homotopy fibers. Nevertheless, altering the last part of the proof slightly, using theorem 0.2 of [12] instead of [7], one obtains the following variant of the main theorem:

THEOREM. *Let* $(f, \alpha, \beta) : (Y, \mathscr{C}, X) \to (Y', \mathscr{C}', X')$. *Assume*:

(i) $N_*(\mathscr{C})$, $N_*(\mathscr{C}')$ *are good simplicial spaces.*

(ii) *For all* $b : B' \to B$, $B({}_B\backslash f, \bar{Y} \times \bar{X}_b) \to Y'(B) \times X'(B')$ *is a fibration with fiber* F.

(iii) *For all* $b : B' \to B$, *the square below induces weak homotopy equivalences on the fibers*:

$$\begin{array}{ccc} B({}_B\backslash f, \bar{Y} \times \bar{X}_b) & \rightarrow & B({}_{B'}\backslash f, \bar{Y} \times \bar{X}_1) \\ \downarrow & & \downarrow \\ Y'(B) \times X'(B') & \rightarrow & Y'(B') \times X'(B') \end{array}$$

Then $B(f, \alpha, \beta)$ *is a quasi-fibration with fiber* F, *up to homotopy equivalence.*

These theorems include as special cases slight variations of all the theorems of §2.

REMARK. If G is a group and X a left G-space, then Morava, [6], defines a category $[X/G]$, whose objects are $x \in X$, and morphisms from x to x' are $g \in G$ such that $gx = x'$. Then $N_*[X/G] \approx B_*(*, G, X)$. He gives credit to G. Segal for this construction.

This suggests that one might attempt to define a (topological) category $[Y, \mathscr{C}, X]$ such that $N_*[Y, \mathscr{C}, X] \approx B_*(Y, \mathscr{C}, X)$ and study $B(f, \alpha, \beta)$ by applying Quillen's Theorem B (generalized to topological categories) to $[Y, \mathscr{C}, X] \to [Y', \mathscr{C}', X']$. This yields a complicated homotopy fiber which can, however, be unraveled to obtain another proof of the main theorem.

This approach will be used in [4] to yield the generalization of the Main Theorem described at the beginning of this section.

Appendix

Analyzing the definition of bar constructions, one sees that they can be defined with very general types of data. All one needs are "coefficient spaces" $Z(v)$ defined for every $v \in V_n(\mathscr{C})$, together with enough structure to define the simplicial operators. In our case, if v is

$$B_0 \xrightarrow{b_1} B_1 \to \cdots \xrightarrow{b_n} B_n,$$

define

$$Z(v) = B({}_{B_n}\backslash f, \bar{Y} \times \bar{X}_v).$$

It suffices to construct $Z(v) \to Z(d_0 v)$, $Z(v) \to Z(d_n v)$, since the other operators are obvious. Now

$$Z(d_0 v) = B({}_{B_n}\backslash f, \bar{Y} \times \bar{X}_{d_0 v}), \qquad Z(d_n v) = B_{B_{n-1}}\backslash f, \bar{Y} \times \bar{X}_{d_n v});$$

$Z(v) \to Z(d_0 v)$ is induced by the natural transformation $\bar{X}_v \Rightarrow \bar{X}_{d_0 v}$ given by b_1, and $Z(v) \to Z(d_n v)$ is induced by the functor ${}_{B_n}\backslash f \to {}_{B_{n-1}}\backslash f$ defined by b_n.

REFERENCES

1. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics **304**, Springer-Verlag, Berlin, 1972.

2. J. P. May, *Classifying spaces and fibrations*, Mem. Am. Math. Soc. **155** (1975).

3. J.-P. Meyer, *Bar and cobar constructions I*, J. Pure Appl. Alg., to appear.

4. J.-P. Meyer, *Bar and cobar constructions II*, in preparation.

5. J. Milnor, *Construction of universal bundles, II*, Ann. of Math. **63** (1956), 430–436.

6. J. Morava, *Hypercohomology of topological categories*, Proceedings, Evanston 1977, Lecture Notes in Mathematics **658**, Springer-Verlag, Berlin, 1978, pp. 383–403.

7. V. Puppe, *A remark on "homotopy fibrations"*, Manuscripta Math. **12** (1974), 113–120.

8. D. G. Quillen, *Higher algebraic K-theory, I*, Lecture Notes in Mathematics **341**, Springer-Verlag, Berlin, 1973, pp. 85–147.

9. G. Segal, *Classifying spaces and spectral sequences*, Pub. Math. I.H.E.S. **34** (1968), 105–112.

10. J. D. Stasheff, *Associated fibre spaces*, Mich. Math. J. **15** (1968), 457–470.

11. R. W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Camb. Phil. Soc. **85** (1979), 91–109.

12. K. A. Hardie, *Quasifibration and adjunction*, Pac. J. Math. **35** (1970), 389–397.

THE JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD 21218 USA